31AH Final Exam Solutions

December 10, 2018

1. Consider the following 3 vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \; v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Compute the following. (i) The unit vector with the same direction as v_2. (ii) The angle between v_1 and v_2. (iii) $v_1 \times v_2$. (iv) The area of the parallelogram spanned by v_2 and v_3. (v) The volume of the parallelepiped spanned by v_1, v_2 and v_3.

Solution.

(i) To normalize a vector, divide it by its length. Thus,

$$\frac{v_2}{|v_2|} = \frac{1}{2} \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ \sqrt{2}/2 \\ 1/2 \end{bmatrix}$$

is the unit vector with the same direction as v_2.

(ii) If u and v are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, then

$$u \cdot v = |u||v| \cos\theta$$

where θ is the angle between u and v. Thus, if θ is the angle between v_1 and v_2,

$$\cos\theta = \frac{v_1 \cdot v_2}{|v_1||v_2|} = \frac{2}{\sqrt{2}\; 2} = \frac{\sqrt{2}}{2}.$$

Therefore, the angle between v_1 and v_2 is $\pi/4$.

(iii)

$$v_1 \times v_2 = \det \begin{bmatrix} \vec{e}_1 & 1 & 1 \\ \vec{e}_2 & 0 & \sqrt{2} \\ \vec{e}_3 & 1 & 1 \end{bmatrix}$$

$$
\begin{aligned}
&= \det\begin{bmatrix} 0 & \sqrt{2} \\ 1 & 1 \end{bmatrix} \vec{e}_1 - \det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{e}_2 + \det\begin{bmatrix} 1 & 1 \\ 0 & \sqrt{2} \end{bmatrix} \vec{e}_3 \\
&= -\sqrt{2}\, \vec{e}_1 - 0\, \vec{e}_2 + \sqrt{2}\, \vec{e}_3 \\
&= \begin{bmatrix} -\sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix}.
\end{aligned}
$$

(iv) The area of the parallelogram spanned by two vectors u and v in $\mathbb{R}^3$ is given by $|u \times v|$. Computing as in part (iii) above we find that

$$
v_2 \times v_3 = \begin{bmatrix} \sqrt{2} - 1 \\ -1 \\ 1 \end{bmatrix}. \tag{1}
$$

Therefore, the desired area is

$$
\left| \begin{bmatrix} \sqrt{2} - 1 \\ -1 \\ 1 \end{bmatrix} \right| = \sqrt{(\sqrt{2} - 1)^2 + (-1)^2 + 1^2} = \sqrt{5 - 2\sqrt{2}}.
$$

(v) The volume of the parallelepiped spanned by 3 vectors v_1, v_2 and v_3 in $\mathbb{R}^3$ is given by $|v_1 \cdot (v_2 \times v_3)|$. Thus, using (1), the desired volume is

$$
\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2} - 1 \\ -1 \\ 1 \end{bmatrix} = \sqrt{2}.
$$

2. Consider the system of linear equations

$$
\begin{aligned}
ax + y + z &= 1 \\
x + ay + z &= 1 \\
x + y + az &= 1.
\end{aligned}
$$

For what values of a does this have (i) a unique solution? (ii) no solution? (iii) infinitely many solutions? (Justify your assertions).

Solution. We first perform a few row operations. Swapping the first two equations gives

$$
\begin{aligned}
x + ay + z &= 1 \\
ax + y + z &= 1 \\
x + y + az &= 1.
\end{aligned}
$$

Now subtract a times the first row from the second row and subtract the first row from the third row to obtain

$$
x + ay + z = 1
$$

$$
\begin{aligned}
(1-a^2)y+(1-a)z =& 1-a \\
(1-a)y+(a-1)z =& 0.
\end{aligned}
$$

There are two cases according as $a = 1$ or $a \neq 1$.

If $a = 1$, then the system reduces to

$$
\begin{aligned}
x+ay+z =& 1 \\
0 =& 0 \\
0 =& 0.
\end{aligned}
$$

which has infinitely many solutions, one for each choice of the nonpivotal variables y and z.

If $a \neq 1$ then we can divide the 2nd and 3rd equations in the system by $1-a$ to obtain

$$
\begin{aligned}
x+ay+z =& 1 \\
(1+a)y+z =& 1 \\
y-z =& 0.
\end{aligned}
$$

There are various quick ways to finish the solution from here. However, we shall rigorously follow the Gaussian elimination algorithm. Swap the 2nd and 3rd rows to obtain

$$
\begin{aligned}
x+ay+z =& 1 \\
y-z =& 0 \\
(1+a)y+z =& 1.
\end{aligned}
$$

Subtract $1+a$ times the 2nd row from the 3rd row to obtain

$$
\begin{aligned}
x+ay+z =& 1 \\
y-z =& 0 \\
(2+a)z =& 1.
\end{aligned}
$$

Therefore, we see that in the case $a \neq 1$ the system is consistent if and only is $a \neq -2$, and in that event, the solution is unique.

Summarizing the observations in the previous paragraphs, the system has a unique solution when $a \neq 1, -2$, it has no solution when $a = -2$, and it has infinitely many solutions when $a = 1$.

3. Find a 2×2 matrix A (with real entries) other than $A = I$ satisfying $A^5 = I$.

Solution. Counterclockwise rotation of the plane about the origin by θ radians is a linear transformation represented by the matrix

$$
A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.
$$

Furthermore, if n is a positive integer, rotation by θ radians n times is the same as rotation by $n\theta$ radians so that

$$A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Hence, if we let A represent rotation by $2\pi/5$ radians, then

$$A^5 = \begin{bmatrix} \cos 5\frac{2\pi}{5} & -\sin 5\frac{2\pi}{5} \\ \sin 5\frac{2\pi}{5} & \cos 5\frac{2\pi}{5} \end{bmatrix} = \begin{bmatrix} \cos 2\pi & -\sin 2\pi \\ \sin 2\pi & \cos 2\pi \end{bmatrix} = I.$$

Remark.

$$A = \frac{1}{4}\begin{bmatrix} \sqrt{6-2\sqrt{5}} & -\sqrt{10+2\sqrt{5}} \\ \sqrt{10+2\sqrt{5}} & \sqrt{6-2\sqrt{5}} \end{bmatrix}.$$

4. Determine whether the following matrix is invertible, and if so, compute its inverse.

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

Solution. A square matrix A is invertible if and only if it row reduces to I. Furthermore, when A is invertible, $[\,A\ \ I\,]$ row reduces to $[\,I\ \ A^{-1}\,]$. Therefore we row reduce $[\,A\ \ I\,]$ to echelon form:

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

subtract row 1 from rows 2 and 3 and add row 1 to row 4

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

multiply rows 2 and 3 by $-\frac{1}{2}$ and then swap them

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1 & 1/2 & -1/2 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

subtract row 2 from row 1 and subtract 2 times row 2 from row 4

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1 & 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 1 & 1 \end{bmatrix}$$

subtract row 3 from row 1 and subtract 2 times row 3 from row 4

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1 & 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 4 & -1 & 1 & 1 & 1 \end{bmatrix}$$

multiply row 4 by 1/4

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1 & 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

subtract row 4 from row 1 and add it to rows 2 and 3

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & -1/4 \\ 0 & 1 & 0 & 0 & 1/4 & 1/4 & -1/4 & 1/4 \\ 0 & 0 & 1 & 0 & 1/4 & -1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 & -1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

It follows that A is invertible and

$$A^{-1} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & -1/4 \\ 1/4 & 1/4 & -1/4 & 1/4 \\ 1/4 & -1/4 & 1/4 & 1/4 \\ -1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

Remark. A one minute solution to this problem is as follows. Observe that

$$A^2 = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4I,$$

or equivalently,

$$A\left(\tfrac{1}{4}A\right) = I.$$

Therefore, A is invertible and $A^{-1} = \frac{1}{4}A$.

5. Let $S = \{v_1, v_2, v_3, v_4, v_5\}$ where

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 5 \\ -1 \\ 5 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 4 \\ -1 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 2 \\ 7 \\ 0 \\ 2 \end{bmatrix}.$$

Find a basis for Span S.

Solution. If we let A denote the 4×5 matrix whose jth column is v_j, then Span $S = \operatorname{img} A$. Consequently, we may exploit Theorem 2.5.4 in the text (page 193), which asserts that a basis for img A is given by the pivotal columns of A. These can be identified by row reduction.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 3 & 5 & 1 & 7 \\ 2 & 1 & -1 & 4 & 0 \\ -1 & 1 & 5 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 & 3 \\ 0 & -1 & -3 & 2 & -4 \\ 0 & 2 & 6 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Evidently, the first, second, and fourth columns of A are pivotal. Therefore, $\{v_1, v_2, v_4\}$ is a basis for Span S.

6. (i) Find a basis for the kernel of

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{bmatrix}.$$

(ii) Based on your calculation in part (i), what can you conclude about the dimension of img A?

Solution. To solve part (i) we follow the recipe provided in Theorem 2.5.6 in the text (page 194).

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -5 & 2 & -3 \\ 0 & -5 & 2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -2/5 & 3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the elements in $\ker A$ are given by choosing the non-pivotal variables x_3 and x_4 arbitrarily, and then solving the equations

$$\begin{aligned} x_1 + 2x_2 \qquad\qquad\quad + x_4 &= 0 \\ x_2 - 2/5x_3 + 3/5x_4 &= 0 \end{aligned}$$

for x_1 and x_2. Furthermore, a basis for $\ker A$ arises from the two choices $x_3 = 1, x_4 = 0$ and $x_3 = 0, x_4 = 1$. When $x_3 = 1$ and $x_4 = 0$, we see that $x_2 = 2/5$ and $x_1 = -4/5$ and when $x_3 = 0, x_4 = 1$, we see that $x_2 = -3/5$ and $x_1 = 1/5$. Thus,

$$\left\{ \begin{bmatrix} -4/5 \\ 2/5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/5 \\ -3/5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\ker A$.

To solve part (ii) recall that the dimension formula (Theorem 2.5.8, page 196) asserts that if A is a general $m \times n$ matrix, then

$$\dim \ker A + \dim \operatorname{img} A = n.$$

In our case, $n = 4$ and we just showed in part (i) that $\ker A$ has a basis consisting of 2 elements, i.e., that $\dim \ker A = 2$. Therefore,

$$\dim \text{img } A = n - \dim \ker A = 4 - 2 = 2.$$

7. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $A = [T]$. Prove that T is one to one if and only if the columns of A are linearly independent.

Solution. We shall use two simple formulas. The first follows immediately the definition of $[T]$.

$$T(\vec{x}) = A\vec{x} \quad \text{ for all } \vec{x} \in \mathbb{R}^n. \tag{2}$$

The second formula, used repeatedly throughout the quarter, follows from the definition of matrix multiplication. If for $j = 1, 2, \ldots, n$, we let $\vec{c}_j$ denote the jth column of A, then

$$A\vec{x} = \sum_{j=1}^{n} x_j \vec{c}_j \quad \text{ for all } \vec{x} \in \mathbb{R}^n. \tag{3}$$

First assume that T is one to one. We wish to prove that the columns of A are linearly independent. Accordingly, assume that $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and

$$\sum_{j=1}^{n} x_j \vec{c}_j = \vec{0}. \tag{4}$$

By (3), this implies that $A\vec{x} = \vec{0}$, which in turn implies via (2) that $T(\vec{x}) = \vec{0}$. Since $T(\vec{0}) = \vec{0}$ as well, and T is assumed to be one to one, it follows that $\vec{x} = \vec{0}$, i.e.,

$$x_j = 0 \text{ for each } j. \tag{5}$$

Summarizing, we have shown that if (4) holds then (5) holds, i.e., the columns of A are linearly independent.

Now assume that the columns of A are linearly independent. We wish to show that T is one to one. Accordingly, assume that $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^n$ and

$$T(\vec{x}) = T(\vec{y}). \tag{6}$$

As T is linear, it follows that

$$T(\vec{x} - \vec{y}) = T(\vec{x}) - T(\vec{y}) = 0.$$

But then (2) implies that $A(\vec{x} - \vec{y}) = 0$, which in turn implies via (3) that

$$\sum_{j=1}^{n} (x_j - y_j)\vec{c}_j = 0.$$

As we assumed that the columns of A are linearly independent, we conclude that $x_j - y_j = 0$ for each j. Therefore,

$$\vec{x} = \vec{y}. \tag{7}$$

Summarizing we have shown that if (6) holds, then (7) holds. Therefore, T is one to one.

8. Show that if A and B are $n \times n$ matrices, then

if A and B are invertible, then AB is invertible.

Is the converse of this statement true? If so, prove it. If not, give a counterexample.

Solution. If A and B are invertible, then

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I.$$

Thus, $B^{-1}A^{-1}$ is a right inverse for AB. Likewise, as

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}B = I,$$

$B^{-1}A^{-1}$ is a left inverse for AB. As AB has both a right and left inverse, AB is invertible.

The converse statement is true. To prove it recall that a square matrix is invertible if and only if it is onto (cf. the discussion surrounding equation 2.2.11 on page 171 of the text). Assume AB is invertible. Then AB is onto. As AB is onto, A is onto. Hence, since A is onto, A is invertible. To see that B is invertible, note that as A and AB are invertible,

$$B = A^{-1}(AB)$$

is a product of invertible matrices, and hence is invertible as well.

9. Diagonalize

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & -1 & -1 \end{bmatrix}.$$

Solution. We first calculate the characteristic polynomial p_A.

$$\begin{aligned} p_A(x) &= \det(xI - A) \\ &= \det \begin{bmatrix} x-1 & -1 & 1 \\ -2 & x & 2 \\ -1 & 1 & x+1 \end{bmatrix} \\ &= (x-1)x(x+1) - 2 + 2 + x - 2(x+1) - 2(x-1) \\ &= (x-1)x(x+1) - 3x \\ &= x^3 - 4x \end{aligned}$$

$$= (x+2)x(x-2)$$

Therefore, the eigenvalues of A are -2, 0, and 2.

For each eigenvalue λ we calculate an eigenvector corresponding to λ by using row reduction to find a nonzero vector in the kernal of

$$\lambda I - A = \begin{bmatrix} \lambda-1 & -1 & 1 \\ -2 & \lambda & 2 \\ -1 & 1 & \lambda+1 \end{bmatrix}.$$

Before picking particular values for λ we can save some time if we do a few row reductions first. Multiply row 3 by -1 and swap it with row 1 to obtain

$$\begin{bmatrix} 1 & -1 & -(\lambda+1) \\ -2 & \lambda & 2 \\ \lambda-1 & -1 & 1 \end{bmatrix}.$$

Now add 2 times row 1 to row 2 and subtract $\lambda - 1$ times row 1 from row 3:

$$\begin{bmatrix} 1 & -1 & -(\lambda+1) \\ 0 & \lambda-2 & -2\lambda \\ 0 & \lambda-2 & \lambda^2 \end{bmatrix}.$$

Finally, subtract row 2 from row 3 to obtain

$$\begin{bmatrix} 1 & -1 & -(\lambda+1) \\ 0 & \lambda-2 & -2\lambda \\ 0 & 0 & \lambda(\lambda+2) \end{bmatrix}. \tag{8}$$

Letting $\lambda = -2$ in (8) we find that v is a corresponding eigenvector provided

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} v = 0,$$

which implies that

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ is an eigenvector for } A \text{ corresponding to } -2. \tag{9}$$

Letting $\lambda = 0$ in (8) we find that v is a corresponding eigenvector provided

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0,$$

which implies that

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ is an eigenvector for } A \text{ corresponding to } 0. \tag{10}$$

Finally, letting $\lambda = 2$ in (8) we find that v is a corresponding eigenvector provided

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 0 & -4 \\ 0 & 0 & 8 \end{bmatrix} v = 0,$$

which implies that

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ is an eigenvector for } A \text{ corresponding to } 2. \tag{11}$$

Combining (9), (10) and (11) yields that $P^{-1}AP = D$ where

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

10. Let P_3 denote the vector space of polynomials of degree ≤ 3, define $D : P_3 \to P_3$ by $D(p(x)) = p'(x)$, and define $T : P_3 \to P_3$ by $T(p(x)) = xp'(x)$. Show that T has an eigenbasis but D does not.

Solution. First observe that for $k = 0, 1, 2, 3$

$$T(x^k) = x(kx^{k-1}) = kx^k,$$

i.e., x^k is an eigenvector for T with corresponding eigenvalue k. Therefore, since $\{1, x, x^2, x^3\}$ is a basis for P_3, $\{1, x, x^2, x^3\}$ is an eigenbasis for T.

That D does not have an eigenbasis follows from the fact that D has only one eigenvalue (namely, 0) of multiplicity 1 (i.e., the dimension of the corresponding eigenspace is 1). To prove this fact assume that λ is an eigenvalue for D with corresponding eigenvector $p(x)$. Since the fourth derivative of a cubic polynomial is 0,

$$\lambda^4 v = D^4 v = 0.$$

Hence, since $v \neq 0$, $\lambda = 0$. This proves that 0 is the only possible eigenvalue for D. To see that the dimension of the corresponding eigenspace is 1 simply note that if $p(x)$ is an eigenvector corresponding to 0, then

$$p'(x) = Dp(x) = 0p(x) = 0,$$

so that p is constant by a well known theorem from calculus. This shows that the eigenspace corresponding to 0 is spanned by the constant polynomial 1. Therefore, its dimension is 1 as claimed.